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A MAGIC CUBE OF SIX.

Probably it can be said with truth that the construction of magic squares and cubes has in itself no immediate utility. Benjamin Franklin, who devised some squares possessing remarkable properties, expresses himself as believing that he might have spent his time to better advantage, and the same thought has been uttered many times by others. As an intellectual recreation, however, and as a means of quickening one's insight into the properties and relations of numbers, this study has real value.

In an admirable work recently published on the subject of *Magic Squares and Cubes,* the author, Mr. W. S. Andrews, after developing very clearly the method of constructing magic cubes of odd numbers and of those divisible by four, passes over the problem of cubes of oddly-even numbers (6, 10, 14, etc.) as not yet solved, though he remarks that he does not believe them mathematically impossible. It was on his suggestion that my attention was turned to the question, and a method soon presented itself of attaining at least a partial solution.

In the first place six magic squares were constructed, exactly similar in plan except that three of them began (at the upper left-hand corner) with odd numbers, each of which was 1 or 1 plus a multiple of 36, and the other three with even numbers, each a multiple of 18. In the first three squares the numbers were arranged in ascending order, in the other three descending. The initial numbers were so chosen that their sum was 651, or $\frac{n}{2}(n^3+1)$, which is the proper summation for each dimension of the projected magic cube. In the construction of these original squares, by the way, the diagrams devised by Mr. Andrews and presented in his book proved a great convenience and saved much time.

Each of the six squares so made is "magic" in that it has the same sum (651) for each column, horizontal row and corner diagonal. As the initial numbers have the same sum the similarity of the squares, with ascending arrangement in one half and descending in the other half, insures the same totals throughout for numbers occupying corresponding cells in the several squares; e. g., taking the third number in the upper row of each square and adding the six together we reach the sum 651, and so for any other position of the thirty-six.

In constructing our cube we may let the original six squares

I

1	215	214	3	212	6
210	8	208	207	11	7
204	203	15	16	14	199
13	17	201	202	200	18
12	206	9	10	209	205
211	2	4	213	5	216

II

198	20	21	196	23	193
25	191	27	28	188	192
31	32	184	183	185	36
186	182	34	33	35	181
187	29	190	189	26	30
24	197	195	22	194	19

III

37	179	178	39	176	42
174	44	172	171	47	43
168	167	51	52	50	163
49	53	165	166	164	54
48	170	45	46	173	169
175	38	40	177	41	180

145	71	70	147	68	150
66	152	64	63	155	151
60	59	159	160	158	55
157	161	57	58	56	162
156	62	153	154	65	61
67	146	148	69	149	72

IV

144	74	75	142	77	139
79	137	81	82	134	138
85	86	130	129	131	90
132	128	88	87	89	127
133	83	136	135	80	84
78	143	141	76	140	73

V

126	92	93	124	95	121
97	119	99	100	116	120
103	104	112	111	113	108
114	110	106	105	107	109
115	101	118	117	98	102
96	125	123	94	122	91

VI

Fig. 1.

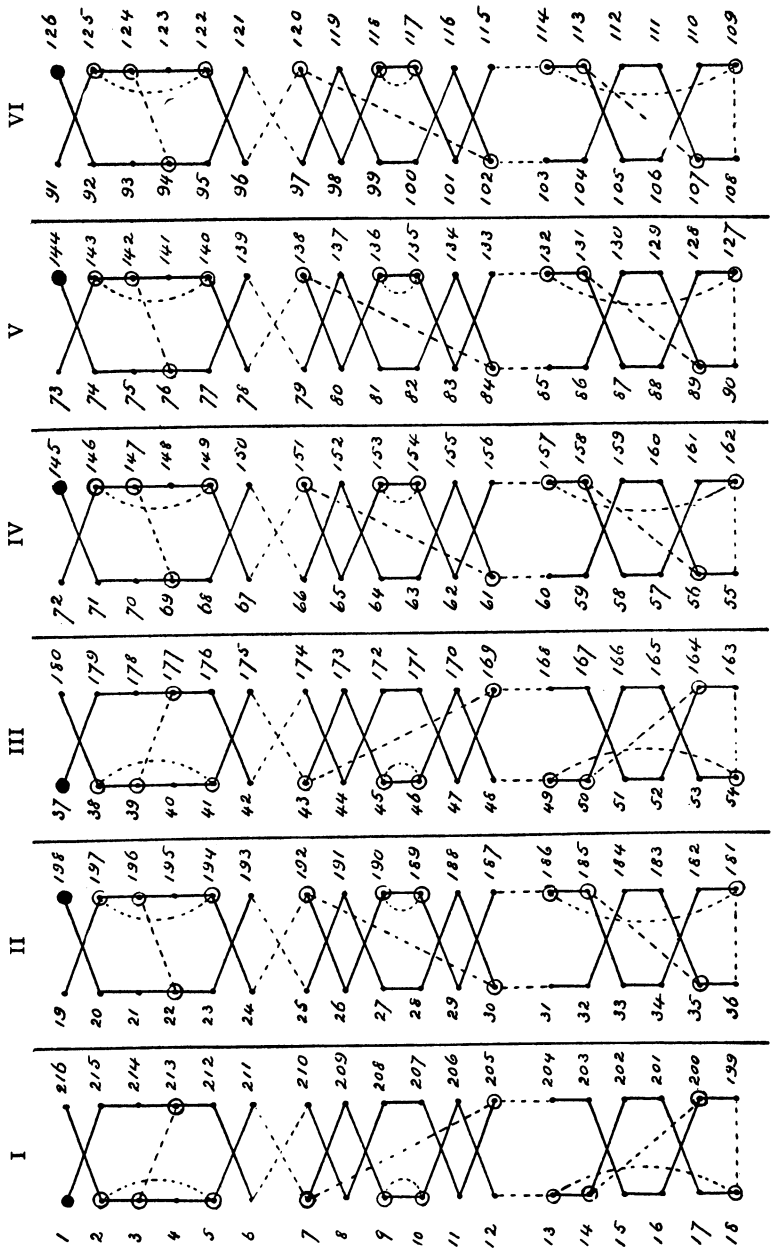
VI
126 125 124 123 122 121 120 119 118 117 116 115 114 113 112 111 110 109
91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108
V
144 143 142 141 140 139 138 137 136 135 134 133 132 131 130 129 128 127
73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90
IV
145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162
72 71 70 69 68 67 66 65 64 63 62 61 60 59 58 57 56 55
III
180 179 178 177 176 175 174 173 172 171 170 169 168 167 166 165 164 163
37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54
II
198 197 196 195 194 193 192 191 190 189 188 187 186 185 184 183 182 181
19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36
I
216 215 214 213 212 211 210 209 208 207 206 205 204 203 202 201 200 199
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18

Fig. 2.

serve as the horizontal layers or strata. We have seen that the vertical columns in the cube must by construction have the correct summation. Furthermore, as the successive right-and-left rows in the horizonal squares constitute the rows of the vertical squares facing the front or back of the cube, and as the columns in the horizontal squares constitute the rows of the vertical squares facing right or left, it is easily seen that each of these twelve vertical squares has the correct summation for all its columns and rows.

Here appears the first imperfection of our cube. Neither the diagonals of the vertical squares nor those of the cube itself have the desired totals, though their *average* footing is correct. It is true further that the footings of the two cubic diagonals originating at opposite extremities of the same plane diagonal average 651, though neither alone is right.

At this point, however, we come upon an interesting fact. While the cubic diagonals vary, the two half-diagonals originating at opposite extremities of either plane diagonal in either the upper or the lower face, and meeting at the center of the cube, together have the sum 651. These correspond in the cube to the "bent diagonals" of Franklin's "square of squares." Of course a moment's reflection will show that this feature is inevitable. The original squares were so constructed that in their diagonals the numbers equidistant from the middle were "complementary", that is, taken together they equaled 217, or $n^3 + 1$, n representing the number of cells in a side of the square. In taking one complementary pair from each of three successive squares to make our "bent diagonal" we must of necessity have $3 \times 217 = 651$.

As in the Franklin squares, so in this cube do the "bent diagonals" parallel to those already described have the same totals. A plane square may be thought of as being bent around a cylinder so as to bring its upper edge into contact with the lower, and when this is done with a Franklin square it will be seen that there is one of these "bent diagonals" for each row. In like manner, if it were possible by some fourth-dimension process analogous to this to set our cube upon itself, we should see that there were six (or in general n) "bent diagonals" for each diagonal in each of the horizontal faces, or 24 in all, and all having the same sum, 651.

The fact that each diagonal in the horizontal squares is made up of three pairs of numbers, each pair having the sum 217, suggests an interesting study. Figure 3 represents a vertical section of the cube in the plane of a diagonal of the upper face. The dotted lines connect numbers, one pair from each of three rows, and in each case the sum of the six numbers is 651. The series represented in the figure—1 119 51 166 98 216, 1 112 8 209 105 216, 1 184 152 65 33 216, 8 126 130 87 91 209, 15 144 119 98 73 202—have each the same total, 651, and the lines connecting the numbers outline some graceful and symmetrical figures. Many more might be drawn, but these examples will illustrate the principle.

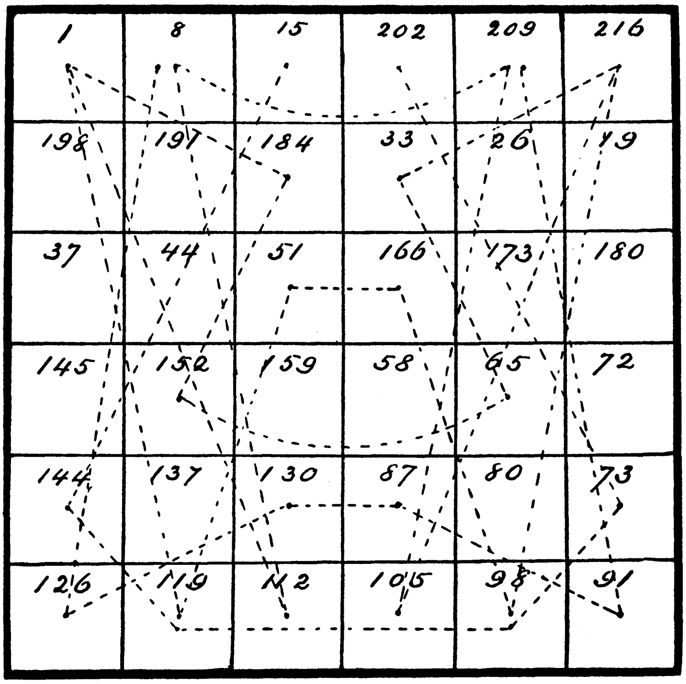

Fig. 3.

Omitting the series described in the last paragraph, which are rather fanciful than natural features of the cube, we may recapitulate the number of occurrences of the characteristic number 651 thus:

In the vertical columns	36	or	n^2
In the rows from front to back	36	or	n^2
In the rows from right to left........	36	or	n^2
In the diagonals of the original squares ..	12	or	$2n$
In the cubic "bent diagonals"	24	or	$4n$
	144	or	$3n^2+6n$

The column of *n* values at the right represents the "general" numbers, found in cubes of 10, 14, etc., as well as in that of 6.

All these characteristics are present no matter in what order the original squares are piled, which gives us 720 permutations. Furthermore, only one form of magic square was employed, and Mr. Andrews has printed diagrams to illustrate at least 128 forms, any one of which might have been used in the construction of our cube. Still further, numerous transpositions within the squares are possible—always provided the vertical totals are guarded by making the same transpositions in two squares, one ascending and the other descending. From this it is easy to see that the numbers 1-216 may be arranged in a very great number of different ways to produce such a cube.

So much for the general arrangement. If we so pile our original squares as to bring together the three which begin with odd numbers and follow them with the others (or *vice versa*) we find some new features of interest. In the arrangement already discussed none of the vertical squares has the correct sum for any form of diagonal. The arrangement now suggested shows "bent diagonals" for the vertical squares facing right and left as follows: Each of the outside squares—at the extreme right or left—has four "bent diagonals" facing the upper and four facing the lower edge. These have their origin in the first, second, fourth and fifth rows moving upward or downward, i. e., in the first two rows of each group—those yielded by original squares starting with odd and those with even numbers. Each of the four inside vertical squares has but two "bent diagonals" facing its upper and two facing its lower edge, and these start in the first and fourth rows—the first of each group of three. This will be true no matter in what order the original squares are piled, provided the odd ones are kept together and the evens together. This will add 32 (8 for each of the two outer and 4 for each of the four inner squares) to the 144 appearances of the sum 651 tabulated above, making 176; but this will apply, of course, only to the cube in which the odd squares are successive and the even squares successive. As the possible permutations of three objects number 6, and as each of these permutations of squares beginning with odd numbers can be combined with any one of the equal number of permutations of the even squares, a total of 36 arrangements is possible.

While the straight diagonals of these squares do not give the required footing the two in each square facing right or left average that sum: thus the diagonals of the left-hand square have totals of 506 and 796, of the second square 708 and 594, third 982 and 320, fourth 596 and 706, fifth 798 and 504, and the right-hand square 986 and 316, each pair averaging 651. I have not yet found any arrangement which yields the desired total for the diagonals, either straight or bent, of the vertical squares facing back or front; nor do their diagonals, like those just discussed, average 651 for any single square, though that is the exact average of the whole twelve.

By precisely similar methods we can construct cubes of 10, 14, 18, and any other oddly-even number, and find them possessed of the same features. I have written out the squares for the magic cube of 10, but time would fail to carry actual construction into higher numbers. Each column and row in the 10-cube foots up 5005, in the 14-cube 19,215, in the 30-cube 405,015, and in a cube of 42 no less than 1,555,869! Life is too short for the construction and testing of squares and cubes involving such sums.

That it is possible to build an absolutely "perfect" cube of 6 is difficult to affirm and dangerous to deny. The present construction fails in that the ordinary diagonals of the vertical squares and of the cube itself are unequal, and the difficulty is made to appear insuperable from the fact that while the proper summation is 651, an odd number, all the refractory diagonals are even in their summation.

The figures which accompany this article were drawn for it by Mr. Andrews, who has taken a lively interest in the cube and its properties. Especially valuable are the diagrams in Figure 2, showing how the numbers of the natural series 1-216 are arranged in the squares which constitute the cube. This is a device of Mr. Andrews's own invention, and certainly is ingenious and beautiful. The diagrams here given for squares of six can be expanded on well-defined principles to apply to those of any oddly-even number, and several of them are printed in the book already mentioned.

It will be noticed that the numbers 1-108 are placed at the left of the diagrams, and those from 109 to 216 inclusive at the right in inverse order. Consequently the sum of those opposite each other is everywhere 217. In each diagram are two pairs of numbers connected by dotted lines and marked ○. These in every case are to be interchanged. Starting then at the heavy dot at the top we follow the black line across to 215, down to 212 (substituting 3 for 213) and back to 6; then across on the dotted line to 210 and along the zigzag black line to 8, 208, 207, 11 and 7 (interchanged with 205);

down the dotted line to 204, then to 203, 15, 16, 14 (in place of 200), 199; then across the diagram and upward, observing the same methods, back to 216. This gives us the numbers which constitute our square No. I, written from left to right in successive rows. In like manner the diagrams in column II give us square No. II, and so on to the end. It is worthy of notice that in the fourth column of diagrams the numbers are written in the reverse of their natural order. This is because it was necessary in writing the fourth square to begin with the number 145 (which naturally would be at the bottom of the diagram) in order to give the initial numbers the desired sum of 651.

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A NEW METHOD FOR MAKING MAGIC SQUARES OF AN ODD DEGREE.

In an endeavor to discover a general rule whereby all forms of magic squares might be constructed, and thereby to solve the question as to the possible number of squares of the fifth order, a method was devised whereby squares may be made, for whose construction the rules at present known to the writer appear to be inadequate.

A *general rule,* however, seems as yet to be unattainable; nor does the solution of the possible number of squares of an order higher than four seem to be yet in sight, though, because of the discovery, so to speak, of hitherto unknown variants, the goal must, at least, have been brought nearer to realization.

The new method now to be described does not pretend to be other than a partial rule, i. e., a rule by which most, but possibly not *all* kinds of magic squares may be made. It is based on De La Hire's method, i. e., on the implied theory that a normal magic square is made up of two primary squares, the one superimposed on the other and the numbers in similarly placed cells added together. This theory is governed by the fact that a given series of numbers may be produced by the consecutive addition of the terms of two or more diverse series of numbers. For example, the series of natural numbers from one to sixteen may be regarded (*a*) as a single series, as stated, or (*b*) as the result of the addition, successively, of all the terms of a series of eight terms to those of another series of two terms. For example, if series No. 1 is composed of 0-1-2-3-4-5-6